Math 4 Honors
Lesson 3-3: *Properties of Rational Functions*

Name [illegible] 2015
Date

Learning Goals

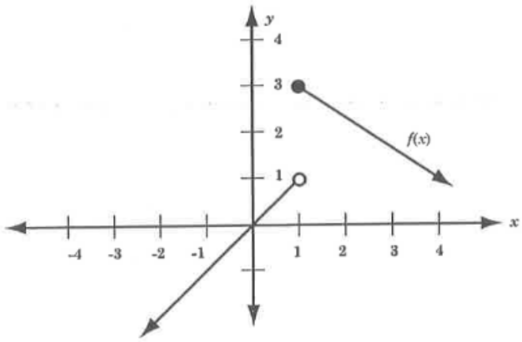
- *I can identify asymptotes (horizontal, vertical, and oblique) for graphs of rational functions.*
- *I can analyze rational functions.*

I. Analyzing Local and Global Behavior of Rational Functions:

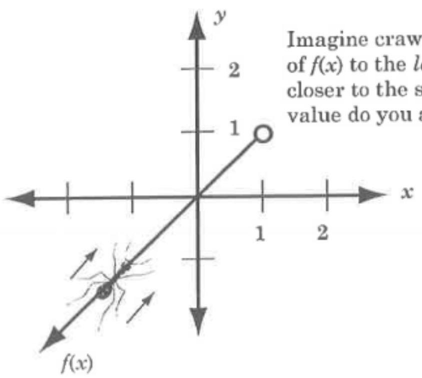
A. Introduction to **Local Behavior** and **Jump Discontinuity**:

- When we observe the **local behavior** of a function $f(x)$ around a specific value $x = p$, we look to see if $f(x)$ approaches a particular value as x approaches p from two directions – the left and the right

Consider the graph of the function $f(x)$ shown below.



- Let's examine the local behavior of the function $f(x)$ around the value $x = 1$.
  - x approaches 1 from the left (Notation: $x \to 1^-$)



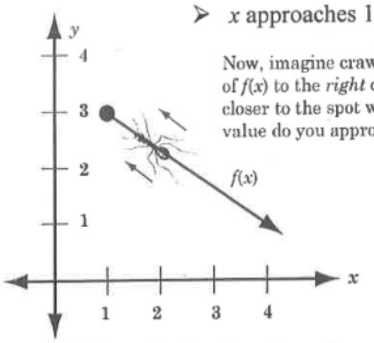
Imagine crawling along the graph of $f(x)$ to the *left* of $x = 1$. As you crawl closer to the spot where $x = 1$, what y value do you approach?

Complete:

$$\lim_{x \to 1^-} f(x) = \underline{\ 1\ }$$

Approach 1 from the left

OVER →



➢ x approaches 1 from the right (Notation: $x \to 1^{+}$)

Approach 1 from the right

Now, imagine crawling along the graph of $f(x)$ to the *right* of $x = 1$. As you crawl closer to the spot where $x = 1$, what y value do you approach?

Complete:

$$\lim_{x \to 1^{+}} f(x) = \underline{\quad 3 \quad}$$

***Since the function doesn't approach the same value from both sides of $x = 1$, we say that $f(x)$ has a **jump discontinuity** at $x = 1$.

B. Local Behavior of Rational Functions

In this section, you will use your graphing calculator to explore **rational functions** - functions of the form $y = \dfrac{f(x)}{g(x)}$, with $f(x)$ and $g(x)$ polynomials.

The table below contains several **rational functions**. Complete the missing values of the table by analyzing the **local behavior** of each function at the location p specified in the table. The first one has been done for you.

***Possible ways to do this: Make a *table* (ctrl T) of values; Make a graph and use the *trace* function to manually enter values around the p-value

Rational Function	p	Behavior of $f(x)$ as $x \to p^{-}$ (from left)	Behavior of $f(x)$ as $x \to p^{+}$ (from right)
$\dfrac{1}{x-3}$	3	$f(x) \to -\infty$	$f(x) \to \infty$
$\dfrac{1}{(x-3)^2}$	3	$f(x) \to +\infty$	$f(x) \to +\infty$
$\dfrac{1}{(x-3)^3}$	3	$f(x) \to -\infty$	$f(x) \to +\infty$
$\dfrac{1}{(x-3)^4}$	3	$f(x) \to +\infty$	$f(x) \to +\infty$
$\dfrac{1}{(x-a)^{2n}},\ n \in \mathbb{N}$	a	$f(x) \to +\infty$	$f(x) \to +\infty$
$\dfrac{1}{(x-a)^{2n-1}},\ n \in \mathbb{N}$	a	$f(x) \to -\infty$	$f(x) \to +\infty$

C. **Practice**

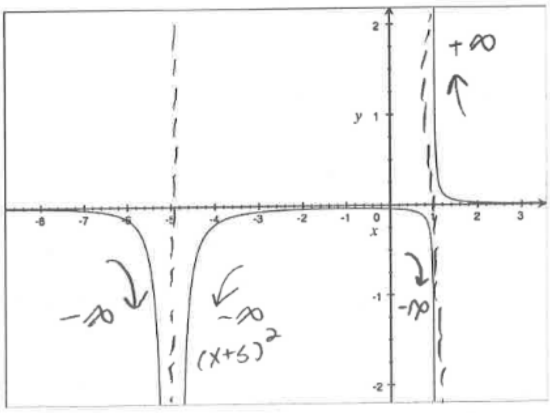
1. Based on the information you compiled in the table, what (if any) generalizations can you make about rational functions of the form $\frac{1}{(x-a)^k}$, $k \in \mathbb{N}$.

The behavior around the asymptotes is like that of the end behavior for the inverse variation functions.

2. Make a prediction about the local behavior of the rational function $f(x) = \frac{1}{x+2}$ near the value $x = -2$. Use your graphing calculator to check your predictions.

$\lim_{x \to -2^-} f(x) = -\infty$ $\qquad$ $\lim_{x \to -2^+} f(x) = \infty$

3. Consider the graph of a rational function of the form $f(x) = \frac{1}{(x-a_1)(x-a_2)(x-a_3)}$ where $a_1, a_2, a_3 \in \mathbb{Z}$. Use the graph to predict the values of a_1, a_2 and a_3.



$a_1 = -5$

$a_2 = -5$

$a_3 = 1$

Check your prediction using graphing features of your calculator.

4. Construct your own rational function $f(x)$ such that $f(x)$ satisfies all 3 of the following properties:

   a. As $x \to 0^+$, $y \to -\infty$
   b. As $x \to 0^-$, $y \to -\infty$
   c. $f(-6) = -1$

y-axis is V.A. / same behavior so even power $\quad y = \frac{K}{x^2}$

$-1 = \frac{K}{(-6)^2}$ $\quad K = -36$

$f(x) = \frac{-36}{x^2}$

OVER →

II. Removable Discontinuities

A. In the previous section, you focused much of your attention on **rational functions** with a constant numerator. **Rational functions** of the form $f(x)=\dfrac{1}{(x-a_1)(x-a_2)\ldots(x-a_n)}$ have **jump discontinuities** at $a_1, a_2, \ldots, a_n$ (i.e. the roots of the denominator).

In this section, we'll examine **rational functions** with linear factors in both the numerator and denominator. As you will see, rational functions such as these may or may not have **jump discontinuities.**

➤ Consider the **rational function** $f(x)=\dfrac{(x-2)\cancel{(x-5)}}{\cancel{(x-5)}}$. Simplify the algebraic expression.

What does this suggest about the graph of $f(x)$? $f(x) = x - 2$

Appears to be linear (discontinuous line)

has a whole

With your graphing calculator, construct a graph of $f(x)$ (prior to simplifying). Your graph should resemble that of the linear function $f(x)=x-2$, *with one small difference*. Complete the following statements to learn more about the rational function $f(x)=\dfrac{(x-2)\cancel{(x-5)}}{\cancel{(x-5)}}$.

a. As $x \to 5^-$, $y \to$ 3

b. As $x \to 5^+$, $y \to$ 3

c. Does $f(x)$ have a **jump discontinuity** at $x=5$? Why or why not?

No. Both approach the same y-value of 3

d. $f(5)=$ $\varnothing$

As you probably discovered, the graph of the **rational function** doesn't "jump" at $x=5$. Since the same y-value is approached from both sides, the function doesn't "jump".

However, since evaluating the function at $x=5$ results in *division by zero* the function itself is *undefined* at $x=5$. (Make a table in your calculator and notice what happens when $x=5$.)

Because the graph breaks at $x=5$, we still say that the function is **discontinuous** at $x=5$. The graph highlights an example of a **removable discontinuity** (some find it helpful to think of "removable" discontinuities as values of x where "holes" occur in the graph of the function).

B. **Practice:**

1. Consider the rational function $f(x)=\dfrac{(x^2-2x-8)}{(x^2+5x+6)}$.

a. Factor the numerator and denominator of the function. At what values of x will the function have a removable discontinuity?

$$f(x)=\frac{(x-4)\cancel{(x+2)}}{\cancel{(x+2)}(x+3)}$$ R.D. when $x=-2$

b. Using the factored form of the numerator and denominator, simplify the function by "cancelling out" like factors. The resulting function should be in the form $\dfrac{(x-a)}{(x-b)}$ with $a \neq b$.

$$f(x)=\frac{x-4}{x+3}$$

Think about it. You can use the "long division algorithm" to write the simplified form of the function in a way that clearly illustrates locations of jump discontinuities. Here's an example using the rational function $g(x)=\dfrac{x-1}{x-2}$.

$$\begin{array}{r} g(x) \\ 1 \\ d(x)\quad x-2\overline{)x-1} \\ x-2 \\ \hline 1 \\ r(x) \end{array}$$

so $\dfrac{x-1}{x-2}=\underbrace{1}_{g(x)}+\dfrac{\overset{r(x)}{1}}{\underset{d(x)}{x-2}}$ there's a jump at $x=2$.

Note that graph of $g(x)$ is actually the graph of $\dfrac{1}{x}$ translated to the right 2 units and translated up 1 unit.

c. Complete similar steps to find the jump discontinuity of the function you wrote in part (b).

$$\begin{array}{r} 1 \\ x+3\overline{)x-4} \\ -(x+3) \\ \hline -7 \end{array}$$

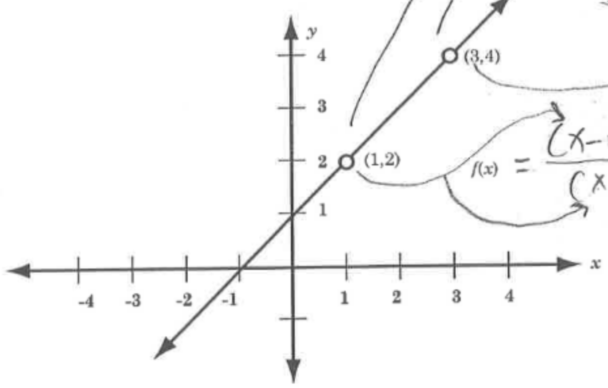
$$\frac{x-4}{x+3}=1+\frac{-7}{x+3}$$

jump when $x=-3$

OVER →

2. Consider the graph below. An equation for the function that generates the graph may be written in the form $f(x) = \dfrac{(x-a)(x-b)(x-c)}{(x-d)(x-e)}$ where $a,b,c,d,e \in$ ¢ (with values not necessarily distinct).

holes: factors must "cancel" out



$$f(x) = \frac{(x-1)(x-3)(x+1)}{(x-1)(x-3)}$$

Find values of $a, b, c, d,$ and e that generate the graph. Check your prediction using graphing features of your calculator. *Make a table of values. Do you see where the holes occur?* Yes!

$a = 1 \quad b = 3 \quad c = -1$

$d = 1 \quad b = 3$

3. Construct your own rational function $f(x)$ such that $f(x)$ satisfies the following 3 properties:

   a. As $x \to 3^-$, $y \to \infty$
   b. As $x \to 3^+$, $y \to \infty$
   c. A **removable discontinuity** exists at $x = 5$.

$\dfrac{K}{(x-3)^2} = y \qquad f(x) = \dfrac{(x-5)}{(x-3)^2(x-5)}$

Challenge: Modify your rational function to satisfy the above three properties **and** the additional property listed below:

$f(2) = 4 \qquad f(x) = \dfrac{K(2-5)}{(2-3)^2(2-5)} = 4$

$\dfrac{K(-3)}{1 \cdot -3} = 4$

$-12 = (-3)K$

$K = 4$

$f(x) = \dfrac{4(x-5)}{(x-3)^2(x-5)}$

Make a table of values. Do you see where the hole occurs? Does $f(2) = 4$?

Yes! Yes!

III. **Essential Discontinuities**

Up to this point we have addressed jump and removable discontinuities. There is one more type of discontinuity when it comes to rational functions . . . . **Essential!**

A. Dig back into your Math 3 Honors archives . . . What is another term for an **essential discontinuity**?

$\boxed{\text{Vertical asymptote!}}$

- Explain what the source of an essential discontinuity is. zeroes of the denominator (after num & denom. have been factored and divided)
- Is it possible for a rational function to have multiple essential discontinuities? Explain by giving an example.

  Yes! $f(x) = \dfrac{x-3}{(x-4)(x-5)(x-6)}$
- Is it possible for a rational function to have zero essential discontinuities? Explain by giving an example.

  Yes! $f(x) = \dfrac{(\cancel{x-5})(x-3)}{(\cancel{x-5})} = x-3$ (only removable)

B. Refer to the function from part B, question 1 on page 5 of this packet.

i. How many essential discontinuities? 1

ii. Write the equation(s) of it(them): $x = -3$

iii. Describe the local behavior around the vertical asymptote using the correct notation.

$\lim_{x \to -3^-} f(x) = -\infty \qquad \lim_{x \to -3^+} f(x) = \infty$

IV. Horizontal and Oblique Asymptotes

A rational function will have either a horizontal or oblique asymptote.

A. Recall the coffee & cream function from the previous investigation: $S(x) = \dfrac{50}{50+x}$

What is the equation of its horizontal asymptote?

$y = 0$

What is the theoretical end behavior of the asymptote?

$\lim_{x \to -\infty} y = 0 = \lim_{x \to \infty} y$

What is the theoretical end behavior of the function?

Same as $y = 0$
(horiz. asymp.)

B. Recall the refrigerator function from the previous investigation: $C(x) = \frac{565 + 72x}{x}$

What is the equation of its horizontal asymptote? $y = 72$

What is the theoretical end behavior of the asymptote?

$\lim_{x \to -\infty} y = 72 = \lim_{x \to \infty} y$

What is the theoretical end behavior of the function?

Same as horiz. asymp., $y = 72$

C. Graph the following function: $f(x) = \frac{x^2 + x + 6}{x - 2}$

$$\begin{array}{r|rrr} 2 & 1 & 1 & 6 \\ & & 2 & 6 \\ \hline & 1 & 3 & 12 \end{array}$$

$y = x + 3$

What type of asymptote? Oblique

What is the equation of the asymptote?

$y = x + 3$

What is the end behavior of the asymptote? positive slope

$\lim_{x \to -\infty} y = -\infty$ $\lim_{x \to \infty} y = \infty$

What is the end behavior of the function?

Same as the oblique asymp., $y = x + 3$

D. Graph the following function: $f(x) = \frac{-2x^2 + 9x - 7}{x - 2}$

$$\begin{array}{r|rrr} 2 & -2 & 9 & -7 \\ & & -4 & 10 \\ \hline & -2 & 5 & 3 \end{array}$$

What type of asymptote? Oblique

What is the equation of the asymptote?

$y = -2x + 5$ negative slope

What is the end behavior of the asymptote?

$\lim_{x \to -\infty} y = \infty$ $\lim_{x \to \infty} y = -\infty$

What is the end behavior of the function?

Same oblique asymp., $y = -2x + 5$

Summary of Horizontal and Oblique Asymptotes:

WITHOUT graphing, how can you predict if a rational function has . . . .

****Hint: Think degrees of the numerator and denominator.*

1. The x-axis for its horizontal asymptote?

   degree of num. < degree of denom.

2. A constant function for its horizontal asymptote?

   degree of num. = degree of denom.

   How do you find the constant value?

   "stack up" Leading coefficients of num & denom

3. An oblique asymptote?

   degree of num. one more than degree of denom.

   How do you find the equation of the oblique asymptote?

   divide

True or false. *If false, explain why (you can use a counterexample if you like).*

1. A rational function can have both a horizontal and oblique asymptote at the same time. False

   These asymptotes determine end behavior. You cannot have ends approaching a constant value and ∞ at the same time.

2. A horizontal asymptote can be intersected by its function. True

3. All rational functions must have either a horizontal an oblique ~~or vertical~~ asymptote. False

   A rational function can simplify to a discontinuous linear function.

4. A rational function can have multiple vertical asymptotes. True

5. Horizontal and oblique asymptotes determine the end behavior of rational functions. True

6. The same value for x can be the source of a hole and a vertical asymptote for a rational function.

   False

   The factor ~~th~~ divide out. $f(x) = \dfrac{x \cancel{-5}}{\quad x-3)}$ — "x=5 is removed" No longer in denom.

OVER →

Lesson 3-3 HW

4. a. $f(x)=\dfrac{4x}{(x-1)(x-1)}$

H.A.: $y=0$

V.A.: $x=1$

b. $g(x)=\dfrac{5x^2+3x+7}{(x-3)(x+3)}$

H.A.: $y=5$

V.A.: $x=3$ $x=-3$

c. $-2\,|\ -3 \quad 1 \quad -12$
$\quad\quad\quad\quad 6$
$\quad\quad -3 \quad 7$

O.A.: $y=-3x+7$

V.A.: $x=-2$

d. $-\frac{1}{2}\,|\ 1 \quad 6.5 \quad 5$
$\quad\quad\quad\quad -.5$
$\quad\quad 1 \quad 6$
$\div 2$

O.A.: $y=\frac{1}{2}x+3$

V.A.: $x=-\frac{1}{2}$

5. a. $f(x)=\dfrac{(x-6)\cancel{(x+2)}}{x\cancel{(x+2)}}$
$=\dfrac{x-6}{x}$

$x\neq 0,-2$

b. $g(x)=\dfrac{(2x+5)\cancel{(x-3)}}{\cancel{(x-3)}(x+3)}$
$=\dfrac{2x+5}{x+3}$

$x\neq -3, 3$

c. $h(x)=\dfrac{\cancel{x}(x-8)\cancel{(x+1)}}{\cancel{x}\cancel{(x+1)}}$
$=x-8$

$x\neq 0,-1$

d. $j(x)=\dfrac{\cancel{x-1}}{(x+5)\cancel{(x-1)}}$
$=\dfrac{1}{x+5}$

$x\neq -5, 1$

6. a. LCD: $(x+2)(x-5)$

$\dfrac{6x(x-5)}{(x+2)(x-5)}+\dfrac{(x-3)(x+2)}{(x+2)(x-5)}=$

$\dfrac{6x^2-30x+x^2-x-6}{(x+2)(x-5)}=$

$\dfrac{7x^2-31x-6}{(x+2)(x-5)}$

b. LCD: $x(2x-1)$

$\dfrac{4(2x-1)}{x(2x-1)}-\dfrac{(3x+7)x}{(2x-1)x}=$

$\dfrac{8x-4-3x^2-7x}{x(2x-1)}=\boxed{\dfrac{-3x^2+x-4}{2x^2-x}}$

c. $\dfrac{x-4}{2x+3}\cdot\dfrac{3x-1}{x^2}=\boxed{\dfrac{3x^2-13x+4}{2x^3+3x^2}}$

d. $\dfrac{5\cancel{x}}{-2x^2+1}\cdot\dfrac{3}{7\cancel{x}}=\boxed{\dfrac{15}{-14x^2+7}}$

8\. a. $3x^2 - 5x - 2 = 0$
$(3x+1)(x-2) = 0$
$\boxed{x = -\frac{1}{3} \quad x = 2}$

b. $x^2 - 8x + 15 = 0$
$(x-5)(x-3) = 0$
$\boxed{x = 3 \quad x = 5}$

c. $0 = \frac{2x+1}{x-4} - \frac{x}{x-1}$
$0 = \frac{(2x+1)(x-1) - x(x-4)}{(x-4)(x-1)}$
$0 = 2x^2 + x - 2x - 1 - x^2 + 4x$
$0 = x^2 + 3x - 1$
Quad Formula:
$\boxed{x = \frac{-3 \pm \sqrt{13}}{2}}$

d. $x^2 - x - 12 = 0$
$(x-4)(x+3) = 0$
$\boxed{x = 4 \quad x = -3}$

e. $-x^2 - x - 2 = 0$
$x^2 + x + 2 = 0$
$\boxed{x = -\frac{1}{2} \pm \frac{i\sqrt{7}}{2}}$

9\. NLA!

a.

$3x+1$	−	+	+
$x-2$	−	−	+
x^2+1	+	+	+
	+	−	+

$-\frac{1}{3}$ $\quad$ 2

$(-\infty, -\frac{1}{3}) \cup (2, \infty)$

b.

$x-5$	−	−	−	+
$x-3$	−	+	+	+
$x-4$	−	−	+	+
	−	+	−	+

$3 \quad 4 \quad 5$

$(-\infty, 3) \cup (4, 5)$

c.

$x-4$	−	−	−	+
$x+3$	−	−	+	+
$x+4$	−	+	+	+
	−	+	−	+

$-4 \quad 3 \quad 4$

$(-4, 3] \cup [4, \infty)$

d.

-1	−	−	−
x^2+x+2	+	+	+
$x+1$	−	+	+
$x-1$	−	−	+
	−	+	

$-1 \quad 1$

$(-1, 1)$

21\. $\boxed{x = -2 \text{ works.}}$ only one solution. $x = -3$ is extraneous b/c it violates domain

23a. $f(x) = \frac{9(\cancel{x-2})}{(x+2)(\cancel{x-2})} = \frac{9}{x+2}$

No zeroes
$f(0) = 4.5$
Hole: $(2, \frac{9}{4})$ (R.D.: $x=2$)
V.A. or E.D.: $x = -2$ $\quad \lim_{x \to -2^-} f(x) = -\infty \quad \lim_{x \to -2^+} f(x) = \infty$
H.A.: $y = 0 \rightarrow \lim_{x \to -\infty} f(x) = 0 = \lim_{x \to \infty} f(x)$

b. $f(x) = \frac{x(x^2-9)}{(x-3)(x-3)} = \frac{x(x+3)(\cancel{x-3})}{(x-3)(\cancel{x-3})}$

Zeroes: $x = 0, -3$
$f(0) = 0$.
V.A. or E.D.: $x = 3 \rightarrow \lim_{x \to 3^-} f(x) = -\infty \quad \lim_{x \to 3^+} f(x) = \infty$
O.A.:
$3 \rfloor \; 1 \quad 3$
$\quad\; 1 \quad 6 \Rightarrow y = x + 6$
$\lim_{x \to -\infty} f(x) = -\infty$
$\lim_{x \to \infty} f(x) = \infty$